A Gibbsian model for message routing in highly dense wireless networks

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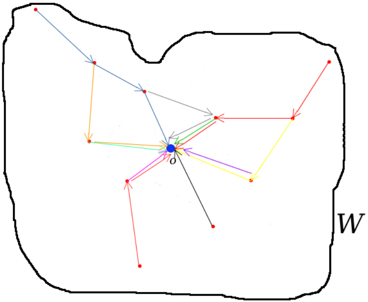
Motivation

Consider a wireless network on a compact communication area $W \subset \mathbb{R}^d$.
Users situated in W randomly, base station $o \in W$.
Idea: assume that each user sends 1 message to the base station.

- Messages travel in hops, possibly using other users as relays. Message trajectories $\to$ straight lines between consecutive steps.
- All users can take at most k_{max} hops, for some $k_{\mathrm{max}} \in \mathbb{N}$ fixed.
- *A priori*, message trajectories are distributed in a uniform way. All trajectories with $1 \leq k \leq k_{\mathrm{max}}$ hops are allowed, even crazy ones.



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- *A priori*, message trajectories are distributed in a uniform way. All trajectories with $1 \leq k \leq k_{\max}$ hops are allowed, even crazy ones.
- We weight this uniform distribution by 2 exponential penalty terms, preferring low interference and little congestion $\to$ Gibbsian trajectory distribution.
  - low interference: high signal-to-interference ratios and not too many hops,
  - little congestion: equal distribution of incoming hops among relays.
- "Common welfare" model, interplay between entropy (probability) and energy (interference+congestion).
- Question: how is the typical behaviour of trajectories (number of hops, length of a hop, shape of a trajectory) in the limit of high density of users?

Distribution of users: Poisson point process

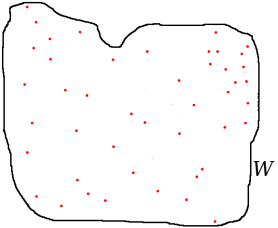
Usual assumption for wireless networks: users form a Poisson point process.

Definition

Let $W \subseteq \mathbb{R}^d$ be bounded and let μ be a finite measure on W. A random collection of points $X = \{X_i\}_{i \in I}$ of W is a Poisson point process (PPP) in W with intensity measure μ, if

(i) $\forall A \subseteq W$ measurable, $\#(X \cap A)$ is Poisson($\mu(A)$)-distributed, i.e., $\mathbb{P}(\#(X \cap A) = n) = \frac{\mu(A)^n}{n!} \mathrm{e}^{-\mu(A)}$, $\forall n \in \mathbb{N}_0$,

(ii) $\forall k \in \mathbb{N}$, for any pairwise disjoint sets $A_1, \dots, A_k \subseteq W$, the random variables $\{\#(X \cap A_i)\}_{i=1}^{k}$ are independent.



Our Gibbsian model

Communication area, users, base station

- $W \subset \mathbb{R}^d$ compact communication area, $\mathrm{Leb}(W) > 0$, $o \in W$ base station (origin of $\mathbb{R}^d$).
- μ finite, absolutely continuous, nonzero measure on W.
- Users: $X^\lambda = \{X_i\}_{i=1}^{N(\lambda)}$ Poisson point process with intensity $\lambda\mu$.
- We assume that $(X^\lambda)_{\lambda>0}$ is such that the empirical measure of users

$$L_\lambda = \frac{1}{\lambda} \sum_{i=1}^{N(\lambda)} \delta_{X_i}$$

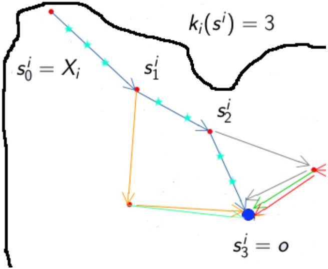
tends to μ almost surely. This holds e.g. if $\lambda \mapsto X^\lambda$ is increasing.
(For $x \in \mathbb{R}^d$, δ_x is a measure on $\mathbb{R}^d$, defined via $\delta_x(A) = 1$ if $x \in A$ and $\delta_x(A) = 0$ otherwise.)

Our Gibbsian model

Message trajectories

- Users: $X^\lambda = \{X_i\}_{i=1}^{N(\lambda)}$ Poisson point process with intensity $\lambda\mu$.
- Fix $k_{\max} \in \mathbb{N}$. Given the users X^λ, the trajectory of the message $X_i \to o$ is random, with a random number of hops in $\{1, \dots, k_{\max}\}$. It has the form

$$s^i = (\underbrace{k_i(s^i)}_{\#\text{hops}};\ \underbrace{s_0^i = X_i}_{\text{transmitter}},\ \underbrace{s_1^i \in X^\lambda, \dots, s_{k_i(s^i)-1}^i \in X^\lambda}_{\text{relays}},\ \underbrace{s_{k_i(s^i)}^i = o}_{\text{receiver}})$$



Our Gibbsian model

Weighting interference

- We choose a path-loss function $\ell\colon [0,\infty) \to (0,\infty)$: continuous, monotone decreasing, describes propagation of signal strength over distance. E.g.: $\ell(r) = \min\{1, r^{-\alpha}\}$, $\alpha > 0$ – Hertzian propagation.
- Signal-to-interference ratio (SIR) of a transmission $X_i \in X^\lambda \to x \in W$:

$$\mathrm{SIR}_\lambda(X_i,\ x,\ X^\lambda) = \frac{\ell(|X_i - x|)}{\frac{1}{\lambda}\sum_{j=1}^{N(\lambda)} \ell(|X_j - x|)}.$$

  The denominator is called the interference at x (rescaled by $1/\lambda$).
- We define a SIR weight term for trajectory collections $s = (s^i)_{i=1}^{N(\lambda)}$:

$$\mathfrak{S}(s) = \sum_{i=1}^{N(\lambda)} \sum_{l=1}^{k_i(s^i)} \mathrm{SIR}_\lambda^{-1}(s^i_{l-1}, s^i_l, X^\lambda).$$

  $\to$ penalty for each step, larger if the SIR is worse (smaller).

Our Gibbsian model

Weighting congestion

- For a trajectory collection s, the number of incoming messages at the user (relay) X_i is $m_i(s) = \sum_{j=1}^{N(\lambda)} \sum_{l=1}^{k_j(s^j)-1} \mathbb{1}\{s_l^j = X_i\}$.
- We define another weight term for the congestion:

$$\mathfrak{M}(s) = \sum_{i=1}^{N(\lambda)} m_i(s)(m_i(s) - 1).$$

$\to$ number of ordered pairs of incoming messages at all relays.
Large penalty for uneven distributions of incoming messages among relays.

The Gibbs distribution

Definition of the Gibbs distribution

For the intensity $\lambda > 0$ and two parameters $\gamma > 0$, $\beta \geq 0$, given the users $X^\lambda = (X_i)_{i=1}^{N(\lambda)}$, the message trajectories are chosen according to the following Gibbs distribution:

$$\mathrm{P}^{\gamma,\beta}_{\lambda,X^\lambda}(s) = \frac{1}{Z^{\gamma,\beta}_\lambda(X^\lambda)} \frac{1}{N(\lambda)^{\sum_{i=1}^{N(\lambda)}(k_i(s^i)-1)}} \exp(-\gamma \mathfrak{S}(s) - \beta \mathfrak{M}(s)).$$

Here $Z^{\gamma,\beta}_\lambda(X^\lambda)$ is the normalizing constant, called partition function, which makes $\mathrm{P}^{\gamma,\beta}_{\lambda,X^\lambda}$ a probability measure:

$$Z^{\gamma,\beta}_\lambda(X^\lambda) = \sum_r \frac{1}{N(\lambda)^{\sum_{i=1}^{N(\lambda)}(k_i(r^i)-1)}} \exp(-\gamma \mathfrak{S}(r) - \beta \mathfrak{M}(r)).$$

Plan to analyze the high-density limit $\lambda \to \infty$

Given $(X^\lambda)_{\lambda>0}$, determine the limiting free energy $\lim_{\lambda\to\infty} \frac{1}{\lambda} \log Z^{\gamma,\beta}_\lambda(X^\lambda)$. The free energy is expected to be given by a variational formula $\to$ minimizer(s) give information about the limiting distribution of trajectories.

First approach to the limiting free energy

Idea: use the empirical measures of message trajectories of given lengths k.
For $k = 1, \ldots, k_{\max}$ and for a trajectory collection $s = (s^i)_{i=1}^{N(\lambda)}$, we put

$$R_{\lambda,k}(s) = \frac{1}{\lambda} \sum_{i=1}^{N(\lambda)} \delta_{(s_0^i, \ldots, s_{k-1}^i)} \mathbb{1}\{k_i(s^i) = k\}.$$

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Properties

- For $k = 1, \ldots, k_{\max}$ and for all s, $R_{\lambda,k}(s)$ is a random element of the set $\mathcal{M}(W^k)$ of finite measures on $W^k = W^{\{0,1,\ldots,k-1\}}$.
- The partition function $Z_\lambda^{\gamma,\beta}(X^\lambda)$ is a function of these measures.
- Each user sends 1 message to o $\Rightarrow$ the 0th marginals $\pi_0 R_{\lambda,k}(s)$ of the $R_{\lambda,k}(s)$'s sum up to the empirical measure of users L_λ:

$$\sum_{k=1}^{k_{\max}} \pi_0 R_{\lambda,k}(s) = \frac{1}{\lambda} \sum_{i=1}^{N(\lambda)} \delta_{X_i} = L_\lambda.$$

- Assumed: $L_\lambda \Rightarrow \mu$, almost surely. Thus, along a subsequence, the $R_{\lambda,k}(\cdot)$'s converge to some $\Sigma = (\nu_k)_{k=1}^{k_{\max}}$, $\nu_k \in \mathcal{M}(W^k)$, with $\sum_{k=1}^{k_{\max}} \pi_0 \nu_k = \mu$.
- The SIR term $\mathfrak{S}(\cdot)$ depends on each $R_{\lambda,k}(\cdot)$ in a continuous, linear way.

The congestion term makes trouble

Problem: the congestion term depends discontinuously on $R_{\lambda,k}(\cdot)$'s as $\lambda \to \infty$.
$\longrightarrow$ No way to express its limit in the terms of the limiting measures $(\nu_k)_{k=1}^{k_{\mathbf{max}}}$.

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$\longrightarrow$ No way to express its limit in the terms of the limiting measures $(\nu_k)_{k=1}^{k_{\mathbf{max}}}$.
Solution: we introduce the empirical measures of users receiving given numbers of incoming messages w.r.t. the trajectory family s:

$$P_{\lambda,m}(s) = \frac{1}{\lambda} \sum_{i=1}^{N(\lambda)} \delta_{X_i} \mathbb{1}\{m_i(s) = m\}, \quad m \in \mathbb{N}_0. \tag{1}$$

Properties

- Each $P_{\lambda,m}(s)$ is a random element of $\mathcal{M}(W)$.
- The congestion term $\mathfrak{M}(\cdot)$ depends linearly on this family of measures:

$$\mathfrak{M}(s) = \sum_{i=1}^{N(\lambda)} m_i(s)(m_i(s) - 1) = \lambda \sum_{m=0}^{\infty} m(m-1) P_{\lambda,m}(s)(W).$$

- Since each user receives exactly m incoming messages for precisely one m, $\sum_{m=0}^{\infty} P_{\lambda,m}(s) = L_\lambda, \quad \forall s, \lambda$.
- So $(P_{\lambda,m}(\cdot))_m$ also converge along a subsequence to some $\Xi = (\mu_m)_{m=0}^{\infty}$, where $\mu_m \in \mathcal{M}(W)$, with $\sum_{m=0}^{\infty} \mu_m = \mu$.

Properties of the limiting measures

Form of limiting measures: $\Psi = (\Sigma, \Xi) = ((\nu_k)_{k=1}^{k_{\mathbf{max}}}, (\mu_m)_{m\in\mathbb{N}_0})$:

- $\nu_k \in \mathcal{M}(W^k)$, $k = 1, \dots, k_{\mathrm{max}}$: limiting distribution of k-hop trajectories,
- $\mu_m \in \mathcal{M}(W)$, $n \in \mathbb{N}_0$: limiting distribution of users (relays) receiving precisely m incoming messages,

Constraints

(i) $\sum_{k=1}^{k_{\mathbf{max}}} \pi_0 \nu_k = \mu$ because each user sends out 1 message to o,

(ii) $\sum_{m=0}^{\infty} \mu_m = \mu$ because each user receives m incoming messages for exactly one m,

(iii) $\sum_{k=1}^{k_{\mathbf{max}}} \sum_{l=1}^{k-1} \pi_l \nu_k = \sum_{m=0}^{\infty} m\mu_m$: the total number of relaying hops of all trajectories $=$ the total number of incoming messages at all relays.

The limiting free energy

Theorem

We have for $\beta \geq 0$, $\gamma > 0$, almost surely w.r.t. the users $(X^\lambda)_{\lambda > 0}$,

$$\lim_{\lambda \to \infty} \frac{1}{\lambda} \log Z_\lambda^{\gamma,\beta}(X^\lambda) = - \inf_{\Psi \text{ satisfying (i),(ii),(iii)}} (\mathrm{I}(\Psi) + \gamma \mathrm{S}(\Psi) + \beta \mathrm{M}(\Psi)).$$

- $\Psi = (\Sigma, \Xi) = ((\nu_k)_{k=1}^{k_{\mathbf{max}}}, (\mu_m)_{m=0}^{\infty})$ satisfying (i)–(iii).
- $\mathrm{S}(\Psi) = \mathrm{S}(\Sigma)$: a limiting SIR term depending only on the ν_k's.
- $\mathrm{M}(\Psi) = \mathrm{M}(\Xi)$: a limiting congestion term depending only on the μ_m's.
- $\mathrm{I}(\Psi)$: an entropy term $\to$ logarithmic rate of combinatorial terms expressing counting complexity. Involves both Σ and Ξ.
- (Precise expressions for $\mathrm{S}, \mathrm{M}, \mathrm{I}$ are on the last slide.)
- We'll see: the variational formula has at least 1 minimizer.

Analysis of the minimizers

Strategy: show that a minimizer exists + all minimizers are positive wherever μ is positive $\to$ identify minimizers via deriving the Euler–Lagrange equations.

Case $\beta > 0$, $\gamma > 0$

Uniqueness is unclear. All minimizers are given in the following implicit way: for $x, x_0, \ldots, x_{k-1} \in W$,

$$\nu_k(\mathrm{d}x_0, \ldots, \mathrm{d}x_{k-1}) = \mu(\mathrm{d}x_0) A(x_0) \prod_{l=1}^{k-1} C(x_l) M(\mathrm{d}x_l) \mathrm{e}^{-\gamma \frac{\int_W \ell(|z-x_l|)\mu(\mathrm{d}x_l)}{\ell(|x_{l-1}-x_l|)}},$$

$$\mu_m(\mathrm{d}x) = \mu(\mathrm{d}x) B(x) \frac{(C(x)\mu(W))^{-m}}{m!} \mathrm{e}^{-\beta m(m-1)}.$$

Here A, B, C are positive functions s.t. (i),(ii),(iii) hold, $M = \sum_{k=1}^{k_{\mathrm{max}}} \sum_{l=1}^{k-1} \pi_l \nu_k = \sum_{m=0}^{\infty} m\mu_m$.

Case $\beta = 0$, $\gamma > 0$

These equations remain true, but they simplify + uniqueness holds. Can write

$$\nu_k(\mathrm{d}x_0, \ldots, \mathrm{d}x_{k-1}) = \mu(\mathrm{d}x_0) A(x_0) \prod_{l=1}^{k-1} \frac{\mu(\mathrm{d}x_l)}{\mu(W)} \mathrm{e}^{-\gamma \frac{\int_W \ell(|z-x_l|)\mu(\mathrm{d}x_l)}{\ell(|x_{l-1}-x_l|)}}.$$

Law of large numbers for the empirical measures of trajectories:
if $\gamma > 0$ and $\beta = 0$, the empirical measures of trajectories $(R_{\lambda,k}((S^i)_{i=1}^{N(\lambda)})_{k=1}^{k_{\mathbf{max}}}$ converge to the unique minimizer $(\nu_k)_{k=1}^{k_{\mathbf{max}}}$ of the variational formula.
This follows from a large deviation principle for these empirical measures.

The minimizer is amenable for analytical investigations $\to$ gives information about the network for high user densities $\lambda < \infty$.
ν_k: limiting distribution of k-hop trajectories.

Qualitative properties of the network

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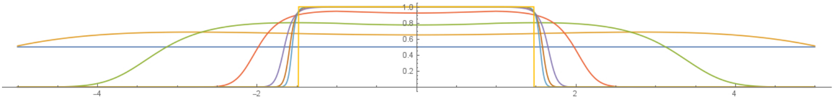

Example: one-hop trajectories in a one-dimensional setting

Density of 1-hop trajectories $\nu_1(\mathrm{d}x)/\mu(\mathrm{d}x)$ for $\gamma = 0, 0.001, 0.01, 0.1, 1, \infty$, for $W = [-5, 5] \subset \mathbb{R}$, $\mu = \mathrm{Leb}|_W$, $o = 0$, $\ell(r) = \min\{1, r^{-4}\}$, $k_{\max} = 2$.
For γ close to 0, ν_1 is almost identically 1/2.
For γ large enough (already for $\gamma = 1$!), $\nu_1(\mathrm{d}x_0)/\mu(\mathrm{d}x_0)$ is close to the indicator function of the 1-hop path being better w.r.t. SIR penalization than any of the 2-hop paths from x_0 to o.

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ν_k: limiting distribution of k-hop trajectories.

1. Typical number of hops in a large-distance limit

Consider larger and larger balls W with users distributed according to Lebesgue measure μ, k_{max} large, transmitter x_0 far out.
Under suitable assumptions on ℓ, the typical length of a hop tends to infinity!
E.g., if $\ell(r) = \min\{1, r^{-\alpha}\}$, $\alpha > d$, then $x_0 \to o$ typically takes $k(|x_0|) \asymp \frac{|x_0|}{\log^{1/\alpha}|x_0|}$ hops of equal length $\asymp \log^{1/\alpha}|x_0|$.
The optimal path follows a straight line with equal-sized hops, macroscopic deviations from it get exponentially unlikely in this limit.

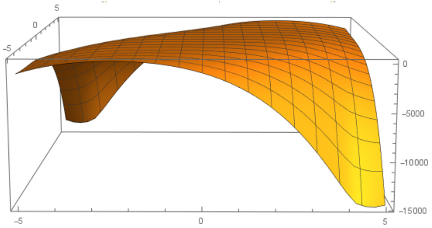
Law of large numbers for the empirical measures of trajectories:
if $\gamma > 0$ and $\beta = 0$, the empirical measures of trajectories $(R_{\lambda,k}((S^i)_{i=1}^{N(\lambda)})_{k=1}^{k_{\max}}$ converge to the unique minimizer $(\nu_k)_{k=1}^{k_{\max}}$ of the variational formula.
ν_k: limiting distribution of k-hop trajectories.

2. Convergence to the straight line for fixed W and large γ

Fix $k_{\max}$ and the communication area $W = \overline{B_r(o)}$, let μ be rotationally symmetric and ℓ strictly monotone increasing. E.g.: $\ell(r) = (1+r)^{-\alpha}$.
Then as $\gamma \to \infty$, we observe convergence to the straight line:
for any $\varepsilon > 0$, $\forall x_0 \in W$, the probability of choosing trajectories $x_0 \to o$ with ≥ 1 hop $\geq \varepsilon$ away from the straight line decays exponentially fast.

Thank you for your attention!

- W. König and A. Tóbiás: A Gibbsian model for highly dense multihop networks. *arXiv:1704.03499* (2017) – for the general case (penalizing interference+congestion).
- W. König and A. Tóbiás: Routeing properties in a Gibbsian model for highly dense multihop networks. *arXiv:1801.04985* (2017/18) – for the applications: qualitative properties of the network, motivation, game-theoretic properties, simulation results.



The limiting entropy, SIR and congestion terms

For $\Psi = ((\nu_k)_{k=1}^{k_{\mathbf{max}}}, (\mu_m)_{m=0}^{\infty})$, the entropy term $\mathrm{I}(\Psi)$ is given as

$$\mathrm{I}(\Psi) = \sum_{k=1}^{k_{\mathbf{max}}} \int_{W^k} \nu_k(\mathrm{d}x_0, \dots, \mathrm{d}x_{k-1}) \log \frac{\mathrm{d}\nu_k}{\mathrm{d}\mu^{\otimes k}}(x_0, \dots, x_{k-1})$$
$$+ \sum_{m=0}^{\infty} \int_W \mu_m(\mathrm{d}x) \log \frac{\mathrm{d}\mu_m}{\mathrm{d}\mu c_m}(x) - \int_W M(\mathrm{d}x) \log \frac{\mathrm{d}M}{\mathrm{d}\mu}(x) - \frac{1}{\mathrm{e}},$$

where c_m are the weights of a Poisson$(\frac{1}{\mathrm{e}\mu(W)})$-distribution.
The expression is to be understood as $+\infty$ if some of the Radon-Nikodym derivatives doesn't exist, and we use the convention $0 \log 0 = 0 \log \frac{0}{0} = 0$.
The limiting SIR term is

$$\mathrm{S}(\Psi) = \sum_{k=1}^{k_{\mathbf{max}}} \sum_{l=1}^{k} \int_{W^k} \nu_k(\mathrm{d}x_0, \dots, \mathrm{d}x_{k-1}) \frac{\int_W \ell(|z - x_l|)\mu(\mathrm{d}z)}{\ell(|x_{l-1} - x_l|)} \in [0, \infty).$$

The limiting congestion term is

$$\mathrm{M}(\Psi) = \sum_{m=0}^{\infty} m(m-1)\mu_m(W) \in [0, \infty].$$