On the Hamilton-Jacobi equation

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Basic Equations in Classical Mechanics

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  Example: $\mathbf{H}(p, x) = T^*(p) + V(x)$

Some Convex Analysis: The Legendre transform

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- Let $A : \mathfrak{X} \to \mathfrak{X}^*$ be a linear symmetric positive operator; and $F(x) = \frac{1}{2}\langle Ax, x \rangle$. Then, $F^*(\xi) = \frac{1}{2}\langle \xi, A^{-1}\xi \rangle$.

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- Let $x_0 \in \mathfrak{X}$. Let

$$F(x) = \chi_{\{x_0\}}(x) = \begin{cases} 0, & \text{if } \; x = x_0 \\ \infty, & \text{if } \; x \neq x_0 \end{cases} .$$

Then $F^*(\xi) = \langle \xi, x_0 \rangle$.

Relation of Lagrangian and Hamiltonian

$$\mathbf{H}(p,x) := \text{Legendre Transform}(\mathbf{L}(x,\cdot))(p) = \sup_{y\in\mathfrak{X}} \left\{ \langle p, y\rangle - \mathbf{L}(x,y) \right\}.$$

The Hamilton-Jacobi equation

Aim: Deriving an equation for the action

$$(x,t) \mapsto S(x,t) = \int_0^t \mathbf{L}(x(t'),\dot{x}(t'))\mathrm{d}t'.$$

Motivation: Starting point for wave-particle duality, QM...

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- First-order partial differential equation
- Highly nonlinear.

How to solve the HJE?

We assume: $\mathbf{L}(x, y) = T(y)$ where

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(Often: $T^*(p) = \frac{p^2}{2m}$.)

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The initial value?
For $x = x_0$: $S(x,t) = 0$ for any $t \geq 0$.
If $x \neq x_0$ and $t \to 0$, we get $S(x,t) \to \infty$. Hence $S_0 = \chi_{\{x_0\}}(x)$.

General initial value problem

Solution method of E. Hopf (1965)

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For $y \in \mathfrak{X}, q \in \mathfrak{X}^*$, we introduce the function

$$v_{y,q}(x,t) = S_0(y) + \langle q, x-y \rangle - tT^*(q),$$

then

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Hence, $v_{y,q}(x,t)$ is a solution of HJE for any y and q.

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Brief remark to Harmonic Anaylsis

Consider the Heat equation on $\mathbb{R}$:

$$\dot{u} = \partial_{xx} u, \quad u(0,x) = u_0(x).$$

How do we solve this equation?
We use Fourier transform

$$\tilde{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{-i\xi x} f(x) \mathrm{d}x.$$

and the L^2-convolution

$$(f * g)(x) = \int_{\mathbb{R}} f(x-y) g(y) \mathrm{d}y.$$

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Heat equation on $\mathbb{R}$:

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Heat kernel - fundamental solution :

$$k_{\mathrm{Heat}}(x, t) = \frac{1}{\sqrt{2t}} \mathrm{e}^{\frac{-x^2}{4t}}.$$

and the solution is

$$u(x, t) = \frac{1}{\sqrt{2\pi}} (k_{\mathrm{Heat}} * u_0)(x, t) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} \mathrm{e}^{\frac{-(x-y)^2}{4t}} u_0(y) \mathrm{d}y.$$

Brief remark to Harmonic Anaylsis

Equations

Initial value problem

Abstract transform

fundamental solution

General solution via Convolution

Brief remark to Harmonic Anaylsis

Equations	**Heat equation**	**Hamilton-Jacobi equation**
Initial value problem	$\dot{u} = \partial_{xx} u$ $u_0(x) = u(0, x)$	$\partial_t S(x,t) = -T^*(\partial_x S(x,t))$ $S(0,x) = S_0(x)$
Abstract transform	Fourier transform $\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{-i\xi x} f(x) \mathrm{d}x$	Legendre transform $\sup_{x \in \mathfrak{X}} \{\langle \xi, x \rangle - F(x)\}$
fundamental solution	Heat kernel $k_{\mathrm{Heat}}(x,t) = \frac{1}{\sqrt{t}} \mathrm{e}^{-\left(\frac{x}{\sqrt{t}}\right)^2}$	Kernel: $k_{\mathrm{HJE}}(x,t) = tT\left(\frac{x}{t}\right)$
General solution via Convolution	L^2-convolution $\int_{\mathbb{R}} k_{\mathrm{Heat}}(x-y) u_0(y) \mathrm{d}y$	inf-convolution $\left(S_0 \overset{\inf}{\Delta} k_{\mathrm{HJE}}\right)(x)$

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have fun in the Botanic Garden!