Cutting a part from many measures

Nevena Palić

6th BMS Student Conference

Equivariant Topological Combinatorics

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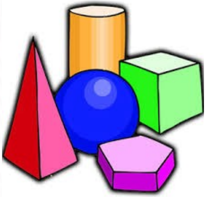

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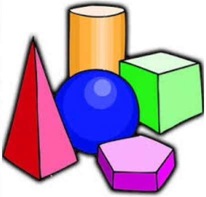

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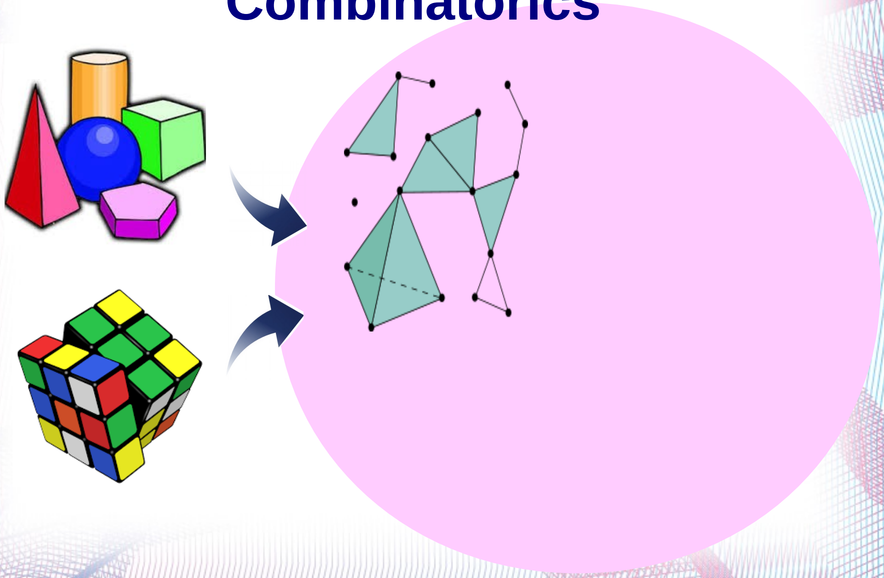

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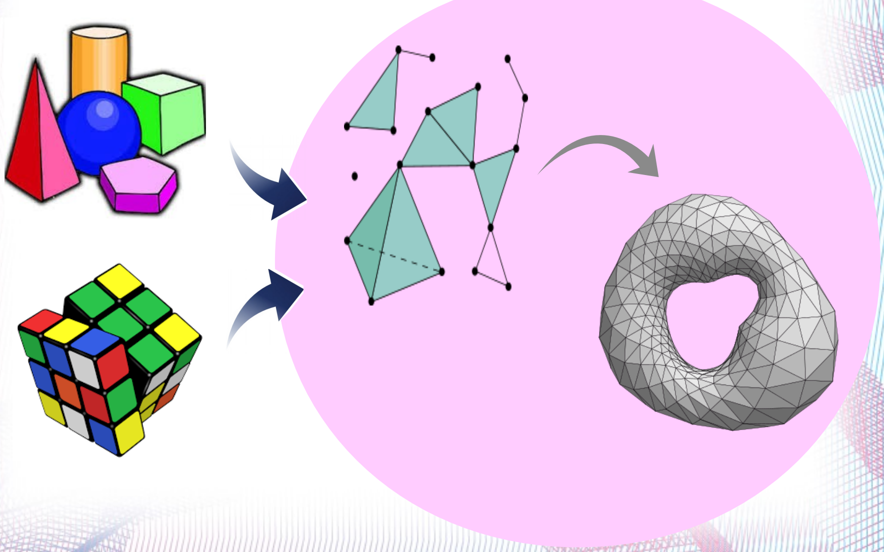

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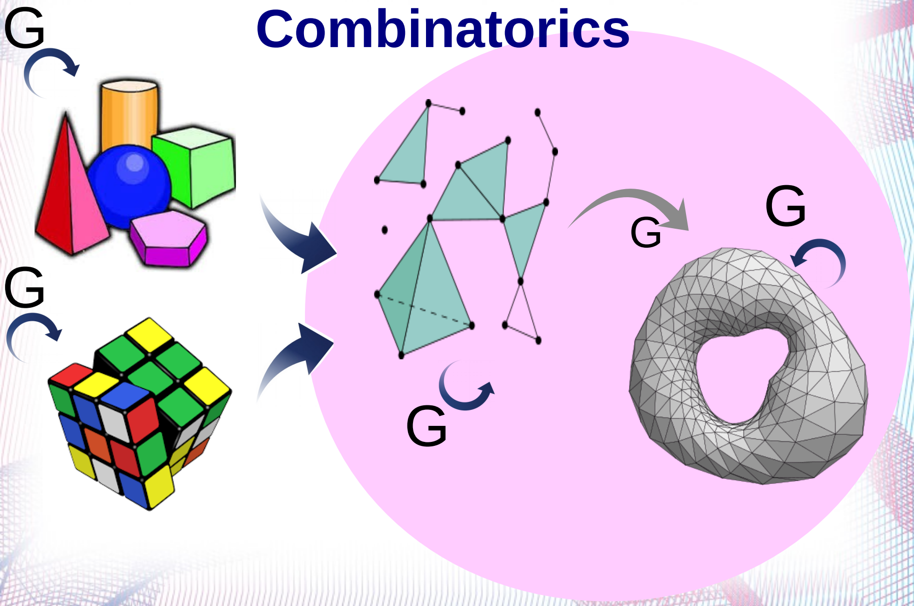

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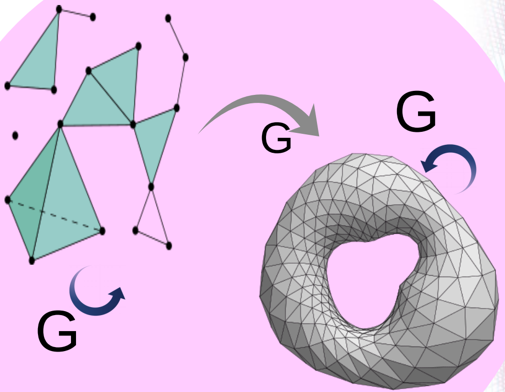

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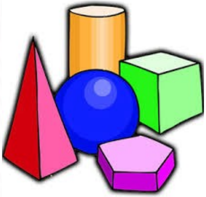
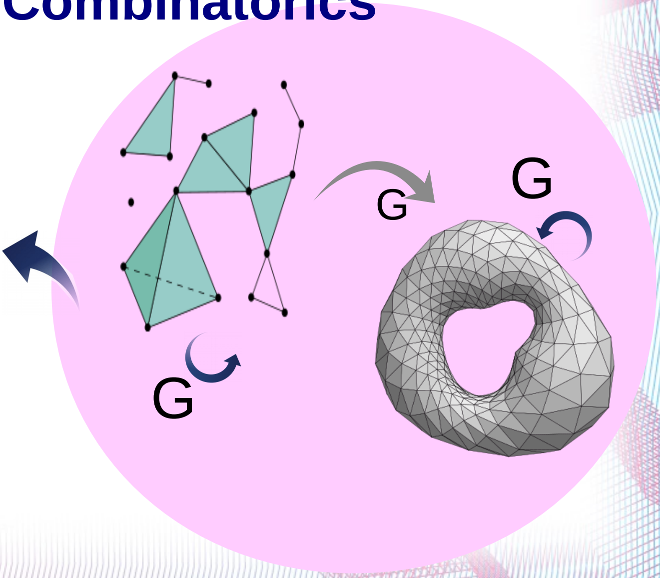




First example

Theorem (Ham Sandwich theorem, Banach 1938)

Any collection of d finite absolutely continuous measures in $\mathbb{R}^d$ can be simultaneously cut into two parts by one hyperplane cut in such a way that each part captures exactly one half of each measure.

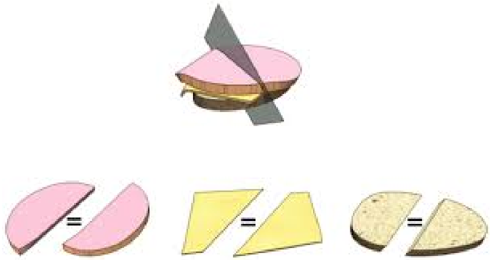
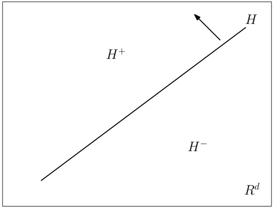


Figure: from Curiosa Mathematica

First example



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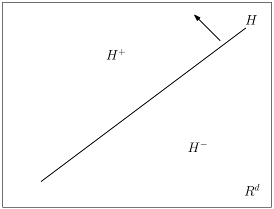

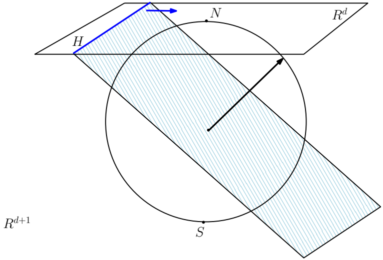

First example

Proof.

$$\begin{array}{rcl} S^d \setminus \{N, S\} & \longrightarrow & \mathbb{R}^d \\ p & \longmapsto & (\mu_1(H_p^+) - \mu_1(H_p^-), \ldots, \mu_d(H_p^+) - \mu_d(H_p^-)) \end{array}$$

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First example

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Assume that there is no solution, i.e., for every hyperplane H there is some $1 \leq i \leq d$ such that $\mu_i(H^+) \neq \mu_i(H^-)$.

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$$S^d \longrightarrow \mathbb{R}^d \setminus \{0\} \longrightarrow S^{d-1}$$

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$$S^d \longrightarrow \mathbb{R}^d \setminus \{0\} \longrightarrow S^{d-1}$$

Group action!

$$S^d \longrightarrow_{\mathbb{Z}_2} S^{d-1}$$

$\square$

First example

Let G be a group that acts on topological spaces X and Y. A map

$$f : X \longrightarrow Y$$

is *G-equivariant* if

$$f(g \cdot x) = g \cdot f(x)$$

for every $x \in X$ and every $g \in G$.

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Theorem (Borsuk-Ulam)

There is no $\mathbb{Z}_2$-equivariant map $S^d \to S^{d-1}$.

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First example – discrete version

Theorem (Discrete Ham Sandwich theorem, Matoušek)

Let $A_1, \ldots, A_d \subset \mathbb{R}^d$ *be disjoint finite point sets in general position. Then there exists a hyperplane* H *that bisects each set* A_i*, such that there are exactly* $\lfloor \frac{1}{2}|A_i| \rfloor$ *points from the set* A_i *in each of the open half-spaces defined by* H*, for every* $1 \leq i \leq d$*.*

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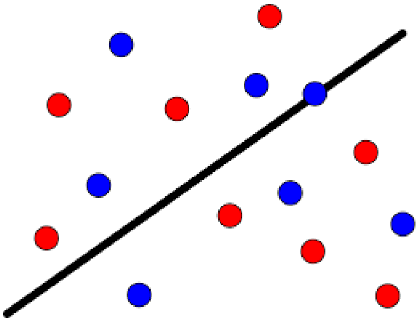


Partitions and measures

Definition

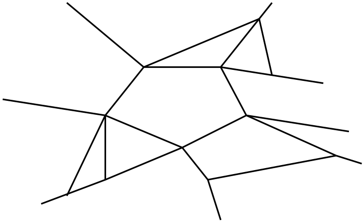
Let $d \geq 1$ and $n \geq 1$ be integers. An ordered collection of closed subsets $(C_1, \ldots, C_n)$ of $\mathbb{R}^d$ is called a *partition* of $\mathbb{R}^d$ if

(1) $\bigcup_{i=1}^{n} C_i = \mathbb{R}^d$,

(2) $\operatorname{int}(C_i) \neq \emptyset$ for every $1 \leq i \leq n$, and

(3) $\operatorname{int}(C_i) \cap \operatorname{int}(C_j) = \emptyset$ for all $1 \leq i < j \leq n$.

A partition $(C_1, \ldots, C_n)$ is called *convex* if all subsets $C_1, \ldots, C_n$ are convex.



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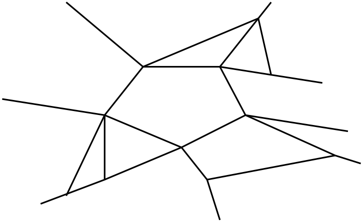

We consider finite absolutely continuous Borel measures.

Motivation

Conjecture (Holmsen, Kynčl, Valculescu, 2017)

- $d \geq 2$, $\ell \geq 2$, $m \geq 2$ and $n \geq 1$ integers,
- $m \geq d$ and $\ell \geq d$,
- $X \subseteq \mathbb{R}^d$, $|X| = \ell n$, X in general position,
- X is colored with at least m different colors.
- a partition of X into n subsets of size ℓ such that each subset contains points colored by at least d colors.

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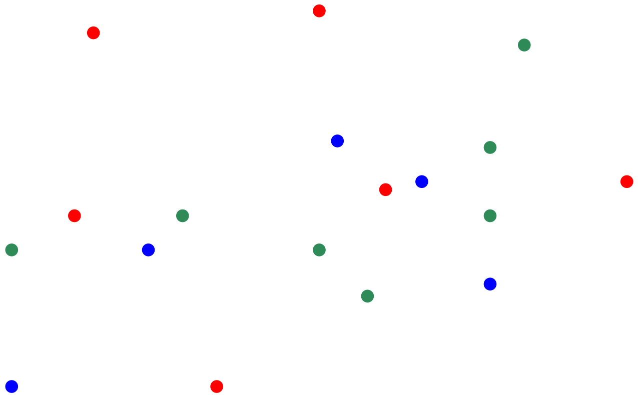
- a partition of X into n subsets of size ℓ such that each subset contains points colored by at least d colors.

Then there exists such a partition of X that in addition has the property that the convex hulls of the n subsets are pairwise disjoint.

Motivation

$d = 2, m = 3, n = 6, l = 3$



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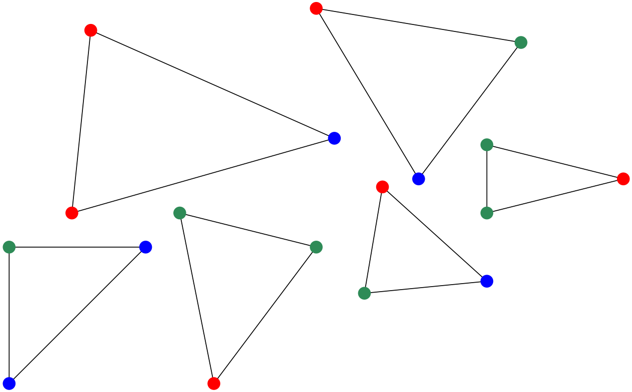

Cutting a part from many measures

Theorem (Blagojević, P., Ziegler, 2017)

- $d \geq 2$, $m \geq 2$, *and* $c \geq 2$ *integers,*
- $n = p^k$ *a prime power,*
- $m \geq n(c-d) + \frac{dn}{p} - \frac{n}{p} + 1$,
- $\mu_1, \ldots, \mu_m$ *positive finite absolutely continuous measures on* $\mathbb{R}^d$.

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Then there exists a convex partition $(C_1, \ldots, C_n)$ such that

$$\#\{j : 1 \leq j \leq m,\ \mu_j(C_i) > 0\} \geq c$$

and

$$\mu_m(C_1) = \cdots = \mu_m(C_n) = \frac{1}{n}\mu_m(\mathbb{R}^d),$$

for every $1 \leq i \leq n$.

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Cutting a part from many measures

Example

- $n = 5$
- $c = 4$
- $d = 3$
- $m = 8$

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CS/TM scheme

$\mathrm{EMP}(\mu_m, n)$ – all convex partitions of $\mathbb{R}^d$ into n convex pieces that equipart the measure μ_m

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\begin{aligned}
f\colon\ \mathrm{EMP}(\mu_m, n) &\longrightarrow \mathbb{R}^{(m-1)\times n} \cong (\mathbb{R}^{m-1})^n \\
(C_1, \dots, C_n) &\longmapsto \begin{pmatrix} \mu_1(C_1) & \mu_1(C_2) & \dots & \mu_1(C_n) \\ \mu_2(C_1) & \mu_2(C_2) & \dots & \mu_2(C_n) \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{m-1}(C_1) & \mu_{m-1}(C_2) & \dots & \mu_{m-1}(C_n) \end{pmatrix}
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\end{aligned}
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f is $\mathfrak{S}_n$-equivariant.

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CS/TM scheme

$$\begin{aligned} V &= \left\{ (y_{jk}) \in \mathbb{R}^{(m-1)\times n} \ : \ \sum_{k=1}^{n} y_{jk} = \mu_j(\mathbb{R}^d) \text{ for every } 1 \le j \le m-1 \right\} \\ &\cong \mathbb{R}^{(m-1)\times(n-1)} \end{aligned}$$

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CS/TM scheme

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$$

$$
\begin{aligned}
f\colon \mathrm{EMP}(\mu_m, n) &\longrightarrow V \subseteq \mathbb{R}^{(m-1)\times n} \\
(C_1, \dots, C_n) &\longmapsto \begin{pmatrix} \mu_1(C_1) & \mu_1(C_2) & \dots & \mu_1(C_n) \\ \mu_2(C_1) & \mu_2(C_2) & \dots & \mu_2(C_n) \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{m-1}(C_1) & \mu_{m-1}(C_2) & \dots & \mu_{m-1}(C_n) \end{pmatrix}
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CS/TM scheme

Arrangement $\mathcal{A} = \mathcal{A}(m, n, c)$ of all subspaces of $V \subset \mathbb{R}^{(m-1)\times n}$ that have at least $m - c + 1$ zeros in some column.

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If there is NO solution, then

$$f : \mathrm{EMP}(\mu_m, n) \longrightarrow_{\mathfrak{S}_n} \bigcup \mathcal{A}$$

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Theorem (Blagojević, P., Ziegler, 2017)

There is no $\mathfrak{S}_n$-equivariant map

$$f : \mathrm{EMP}(\mu_m, n) \longrightarrow_{\mathfrak{S}_n} \bigcup \mathcal{A}.$$

Proof of the topological result

We construct these $\mathfrak{S}_n$-equivariant maps:

$$\begin{tikzcd}
\mathrm{EMP}(\mu_m, n) \arrow[r, "f"] & \bigcup \mathcal{A} \arrow[r, "\beta"] & \operatorname{hocolim}_{P(\mathcal{A})} \mathcal{C} \arrow[r, "\gamma"] & \operatorname{hocolim}_{Q'} \mathcal{D} \arrow[r, "\delta"] & \operatorname{hocolim}_{Q'} \mathcal{E} \arrow[d, "\eta"] \\
\mathrm{Conf}(\mathbb{R}^d, n) \arrow[u, "\alpha"] \arrow[rrrr, dashed, "g := \eta\circ\delta\circ\gamma\circ\beta\circ f\circ\alpha"] & & & & X
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\end{tikzcd}$$

$$\mathrm{Conf}(\mathbb{R}^d, n) := \{(x_1, \ldots, x_n) \in (\mathbb{R}^d)^n \mid x_i \neq x_j \text{ for all } 1 \leq i < j \leq n\}$$

References

[1] Pavle V.M. Blagojević, Nevena Palić and Günter M. Ziegler, *Cutting a part from many measures*, arXiv:1710.05118.

[2] Andreas F. Holmsen, Jan Kynčl and Claudiu Valculescu, *Near equipartitions of colored point sets*, Computational Geometry **65** (2017), 35–42.

[3] Jirží Matoušek, *Using the Borsuk-Ulam theorem*, Lectures on Topological Methods in Combinatorics and Geometry, Springer Publishing Company, Incorporated 2007.

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