(One Small Aspect of) Convexity and Curvature

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- Jordan-Brouwer tells us that Σ bounds a domain $\Omega \subset \mathbb{R}^{n+1}$
- Ω is convex if for every $x, y \in \Omega$, the straight line connecting x and y remains in Ω

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This is useful for doing analysis (picture A solving some PDE).

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If $\{\nu_\alpha\}$ is an orthonormal basis for $N\Sigma$, we define

$$\begin{aligned} A_\alpha : T\Sigma \times T\Sigma &\to \mathbb{R} \\ (X, Y) &\mapsto \langle D_X \nu_\alpha, Y\rangle \end{aligned}$$

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Convexity no longer makes sense, but maybe we can place a (nice) condition on A which recovers some of its properties...

Nice = basis invariant.

Some basis invariant objects:

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- $\operatorname{tr} A = \sum_\alpha \operatorname{tr} A_\alpha$
- $|\operatorname{tr} A|^2$

Ansatz for the new condition:

$$|A|^2 \leq c_n |\operatorname{tr} A|^2.$$

Returning to $k = 1$, $A : T\Sigma \times T\Sigma \to \mathbb{R}$ is symmetric and can be diagonalised. Denote its eigenvalues by $\lambda_1, \ldots, \lambda_2$.

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  in which case $\operatorname{tr} A \geq 0 \Rightarrow A \geq 0$.
- $\operatorname{tr} A \geq \Rightarrow A \geq 0$ still holds if $c_n \leq \frac{1}{n-1}$ (sharp).

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for larger values of k? What properties does it share with convexity?

For $k = 1$ and $n = 2$, $A \geq 0$ implies that the Gauss curvature of Σ (given by $\det A$) is nonnegative.

The Gauss curvature is intrinsic (depends only on the metric structure of Σ) and vanishes if and only if Σ is locally isometric to $\mathbb{R}^2$.

For $n \geq 2$, the Riemannian curvature operator

$$\mathcal{R} : \wedge^2 T\Sigma \times \wedge^2 T\Sigma \to \mathbb{R}$$

measures intrinsic curvature (vanishes if and only if Σ is locally isometric to $\mathbb{R}^n$).

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For $k = 1$ and $n \geq 2$, $A \geq 0$ if and only if $\mathcal{R} \geq 0$.

For all $k \geq 1$, we now know that if $|\operatorname{tr} A| > 0$ then

$$|A|^2 \leq \frac{1}{n-1} |\operatorname{tr} A|^2 \Rightarrow \mathcal{R} \geq 0.$$

To summarise, $|A|^2 \leq \frac{1}{n-1}|\operatorname{tr} A|^2$ is :

- Extrinsic
- Independent of choices of bases
- Forces nonnegativity of the intrinsic curvature,

and so, might serve as a high-codimension substitute for convexity.

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For example...

A Liouville-type theorem for high codimension mean curvature flow:

Theorem

Let $\Sigma_t^n \subset \mathbb{R}^{n+k}$, $n \geq 2$, solve mean curvature flow for all $t \in (-\infty, 0)$. If there is a positive constant ε such that

$$|A|^2 - \frac{1}{n-1}|\operatorname{tr} A|^2 \leq -\varepsilon|\operatorname{tr} A|^2 < 0$$

on Σ_t for all $t \in (-\infty, 0)$, then Σ_t is a family of homothetically shrinking spheres inside an affine copy of $\mathbb{R}^{n+1}$.