COMPLEX UNIFORMIZATION OF FERMAT CURVES

Pilar Bayer
University of Barcelona

BMS Student Conference
Berlin Mathematical School, 2018-02-22

based on joint work with Jordi Guàrdia

References

1. Bayer, P.; Guàrdia, J.: Hyperbolic uniformization of Fermat curves. *Ramanujan J.* 12 (2006), no. 2, 207–223.
2. Bayer, P.: Uniformization of certain Shimura curves. *Differential Galois theory* (Bedlewo, 2001), 13–26, Banach Center Publ., 58, Polish Acad. Sci. Inst. Math., Warsaw, 2002.
3. Guàrdia, J.: A fundamental domain for the Fermat curves and their quotients. Contributions to the algorithmic study of problems of arithmetic moduli. *Rev. R. Acad. Cienc. Exactas Fís. Nat.* 94 (2000), no. 3, 391–396.

Outline

1. Curves and Riemann surfaces
2. Fermat curves
3. The Fermat sinus and cosinus functions: $(\mathrm{sf}, \mathrm{cf})$
4. Fermat tables

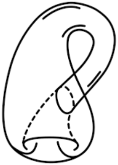
Outline

Compact surfaces

Theorem

Any connected compact surface is homeomorphic to:

1. The sphere ($abb^{-1}a^{-1}$).
2. The connected sum of g tori ($aba^{-1}b^{-1}$), for $g \geqslant 1$.
3. The connected sum of k real projective planes ($abab$), for $k \geqslant 1$.



$$P \# P = K = abab^{-1}$$

Compact connected orientable surfaces

$g = 0$

$1 \leqslant g \leqslant 3$

Image source: Henry Segerman

Compact connected orientable surfaces are classified by their genus g.

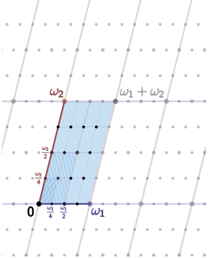
$g = 1$. Weierstrass's elliptic functions

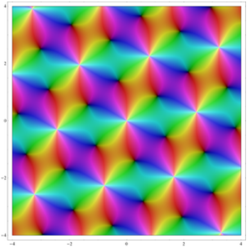
$$\Lambda = \{m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\} \subseteq \mathbb{C}, \quad \tau = \omega_2/\omega_1, \quad \Im(\tau) > 0$$

$$\wp(z; \omega_1, \omega_2) = \frac{1}{z^2} + \sum_{n^2+m^2\neq 0} \left\{ \frac{1}{(z+m\omega_1+n\omega_2)^2} - \frac{1}{(m\omega_1+n\omega_2)^2} \right\}$$

$$E_\Lambda : \wp'^2(z) = 4\wp^3(z) - g_2\wp(z) - g_3, \quad g_2(\omega_1, \omega_2), g_3(\omega_1, \omega_2) \in \mathbb{C}$$

$$\mathbb{C}/\Lambda \simeq E_\Lambda(\mathbb{C}), \quad z \mapsto (\wp(z), \wp'(z))$$





$$Y^2 = 4X^3 - g_2X - g_3, \quad \Delta = g_2^2 - 27g_3^2 \neq 0$$

Hyperbolic geometry

$\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$, complex projective line, Riemann sphere

$\alpha = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{SL}(2, \mathbb{C}),\ \alpha(z) = \dfrac{az+b}{cz+d}$, Möbius transformations

$\mathbf{PSL}(2, \mathbb{C}) = \mathbf{SL}(2, \mathbb{C})/\{\pm \mathrm{I}_2\}$, conformal transformations of $\mathbb{P}^1(\mathbb{C})$

A model for the hyperbolic plane:

$$\mathcal{H} = \{z = x + iy \in \mathbb{C} : y > 0\}$$

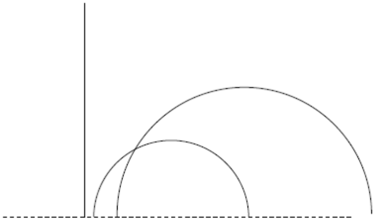
$$\mu = \frac{dx^2 + dy^2}{y^2}, \quad d(z_1, z_2) = \left| \operatorname{arc\,cosh} \left(1 + \frac{|z_1 - z_2|^2}{2 z_1 z_2} \right) \right|$$

$\mathbf{PSL}(2, \mathbb{R}) = \mathbf{SL}(2, \mathbb{R})/\{\pm \mathrm{I}_2\}$, hyperbolic motions of $\mathcal{H}$

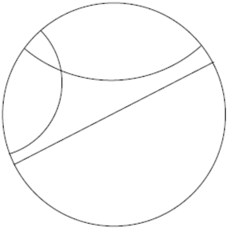
The Poincaré disk model for the hyperbolic plane

$$\mathcal{D}_r = \{z \in \mathbb{C} : z\bar{z} < r^2\}, \quad r \in \mathbb{R},\ r > 0$$
$$\mathrm{Aut}(\mathcal{D}_r) \simeq \left\{ \begin{bmatrix} a & br \\ \bar{b}/r & \bar{a} \end{bmatrix} : a, b \in \mathbb{C},\ |b| < |a| \right\} /\mathbb{R}^*.$$

$$\mathcal{H} \simeq \mathcal{D}_r, \quad z = x + iy \mapsto z = r\left(\frac{2x}{x^2 + (1+y)^2}, \frac{x^2 + y^2 - 1}{x^2 + (1+y)^2} \right)$$



geodesics in $\mathcal{H}$



geodesics in $\mathcal{D}_r$

Types of conformal isometries of $\mathcal{H}$

$$\alpha = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{SL}(2, \mathbb{R}), \quad \alpha \neq \pm \mathrm{I}_2. \text{ Fixed points:}$$

$$\alpha(z) = z \Leftrightarrow z = \frac{a - d \pm \sqrt{(a+d)^2 - 4}}{2c}$$

hyperbolic:	$\lvert \mathrm{tr}(\alpha) \rvert > 2$,	$\mathbb{P}^1(\mathbb{C})^\alpha = \{z_1, z_2\}$,	$z_1, z_2 \in \mathbb{P}^1(\mathbb{R})$
elliptic:	$\lvert \mathrm{tr}(\alpha) \rvert < 2$,	$\mathbb{P}^1(\mathbb{C})^\alpha = \{z, \overline{z}\}$,	$z \in \mathcal{H}$, elliptic
parabolic:	$\mathrm{tr}(\alpha) = \pm 2$,	$\mathbb{P}^1(\mathbb{C})^\alpha = \{z\}$,	$z \in \mathbb{P}^1(\mathbb{R})$, cusp

Conjugacy classes in $\mathbf{SL}(2, \mathbb{R})$:

$$\begin{bmatrix} \lambda & \\ & \lambda^{-1} \end{bmatrix}, \lambda \neq 1; \quad \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \theta \in \mathbb{R} \setminus 2\pi\mathbb{Z}; \quad \begin{bmatrix} \pm 1 & h \\ 0 & \pm 1 \end{bmatrix}$$

Fuchsian groups and Riemann surfaces

$\Gamma \subseteq \mathbf{SL}(2,\mathbb{R})$ discrete subgroup, $\overline{\Gamma} \subseteq \mathbf{PSL}(2,\mathbb{R})$

P_Γ set of cusps, $\mathcal{H}^* = \mathcal{H} \cup \mathrm{P}_\Gamma$, $\pi: \mathcal{H}^* \to \Gamma\backslash\mathcal{H}^*$

$\Gamma\backslash\mathcal{H}^* \simeq X(\Gamma)(\mathbb{C})$ compact Riemann surface

$$\sharp\overline{\Gamma}_z = \begin{cases} \infty & \text{if } z \text{ is a cusp} \\ e_{\pi(z)} > 1 & \text{if } z \text{ is elliptic} \\ 1 & \text{otherwise} \end{cases}$$

$\Gamma' \subseteq \Gamma$, $[\overline{\Gamma} : \overline{\Gamma}'] = n$, $\varphi: X(\Gamma') \to X(\Gamma)$, $e_{w,\varphi} = [\overline{\Gamma}_{\varphi(w)} : \overline{\Gamma}'_w]$

Hurwitz formula: $2g' - 2 = n(2g-2) + \sum_{w \in X(\Gamma')} (e_{w,\varphi} - 1)$

Laszlo Fuchs obtained his PhD in 1858 under Ernst Kummer in Berlin.

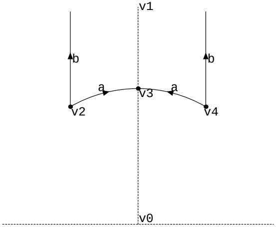
Fundamental domains

Γ Fuchsian group, $\mathcal{F} \subseteq \mathcal{H}$ connected domain

(i) $\mathcal{H} = \bigcup_{\gamma \in \Gamma} \gamma(\mathcal{F})$,

(ii) $\mathcal{F} = \overline{U}$, U open set, $U = \mathrm{int}(\mathcal{F})$,

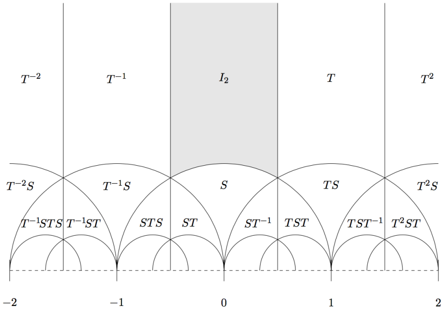
(iii) $\gamma(U) \cap U = \emptyset$, for any $\gamma \in \Gamma$, $\gamma \neq \pm 1$.

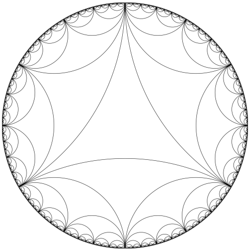


$\mathbf{SL}(2, \mathbb{Z}) = \langle S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \rangle$

fundamental domain for the modular group

Hyperbolic tesselations by $\mathbf{SL}(2,\mathbb{Z})$





Γ-Automorphic forms

$$\alpha = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{GL}^+(2,\mathbb{R}), \quad k \in \mathbb{Z}, \quad j(\alpha, z) := cz + d$$

$$f : \mathcal{H} \to \mathbb{P}^1, \quad (f|_k\alpha)(z) = \det(\alpha)^{k/2} j(\alpha, z)^{-k} f(\alpha z), \quad z \in \mathcal{H}$$

Definition

A meromorphic function $f(z)$ on $\mathcal{H}$ is called a *Γ-automorphic form of weight k* if it is meromorphic at all cusps and satisfies $f|_k\gamma = f$, for all $\gamma \in \Gamma$.

$$\mathcal{A}_{2m}(\Gamma) \simeq \Omega^m(X(\Gamma)), \quad f \mapsto \omega_f, \quad f(z)(dz)^m = \omega_f \circ \pi$$

$$\boxed{\mathcal{A}_0(\Gamma) = \mathbb{C}(X(\Gamma)) \quad \text{field of } \Gamma\text{-automorphic functions}}$$

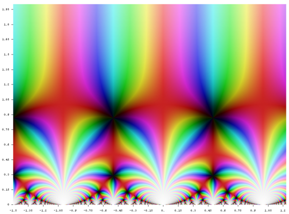
$g = 0$. Klein's j invariant

$\Gamma = \mathbf{SL}(2,\mathbb{Z})$ modular group, $\mathbb{C}(X(\mathbf{SL}(2,\mathbb{Z}))) = \mathbb{C}(j)$

$g_2 = 60 \sum_{(m,n)\neq(0,0)} (m+nz)^{-4}$ modular form of weight 4

$g_3 = 140 \sum_{(m,n)\neq(0,0)} (m+nz)^{-6}$ modular form of weight 6

$$\Delta = g_2^3 - 27 g_3^2 \in S_{12}(\Gamma), \quad j(z) = 1728 \frac{g_2^3}{\Delta} \in \mathcal{A}_0(\Gamma), \qquad z \in \mathcal{H}$$



$$j(q) = \frac{1}{q} + 744 + 196\,884 q + O(q^2)$$

$q = e^{2\pi i z}$ local parameter, $\quad j(e^{\frac{2\pi i}{3}}) = 0, \quad j(i) = 1728$

Schwarzian derivatives

Theorem

(a) The derivative f' of an automorphic function f is an automorphic form of weight 2.

(b) If f is an automorphic form of weight k, then

$$kff'' - (k+1)(f')^2$$

is an automorphic form of weight $2k+4$.

Definition

The Schwarzian derivative with respect to z, $\{w, z\}$, of a non-constant smooth function $w(z)$ is defined by

$$\{w, z\} = \frac{2w'w''' - 3(w'')^2}{4(w')^2}, \quad w' = \frac{dw}{dz}.$$

Hermann Schwarz obtained his PhD in 1864 under Kummer and Weierstrass in Berlin.

Automorphic derivatives

Definition

The Γ-automorphic derivative $\{w, z\}_\Gamma$ of a non-constant smooth function $w(z)$ with respect to z is defined by

$$\{w, z\}_\Gamma := \frac{\{w, z\}}{w'^2}, \quad w' = \frac{dw}{dz}.$$

Proposition

If $w(z)$ is a Γ-automorphic function on $\mathcal{H}$, so is

$$\{w, z\}_\Gamma = \frac{2w'w''' - 3(w'')^2}{4(w')^4} = -\{z, w\}.$$

That is: $\{w, \gamma(z)\}_\Gamma = \{w, z\}_\Gamma$, for all $\gamma \in \Gamma$.

Connection with second order linear differential equations

Theorem (Poincaré)

Let Γ be a Fuchsian group of the first kind, $w(z) \in \mathcal{A}_0(\Gamma)$ a non-constant automorphic function and $\zeta(w)$ be its inverse function. Then

$$\zeta(w) = \frac{\eta_1(w)}{\eta_2(w)},$$

where $\{\eta_1, \eta_2\}$ is a fundamental system of solutions of the ordinary differential equation

$$\frac{d^2\eta}{dw^2} = \{w, z\}_\Gamma \, \eta.$$

Moreover, $\{w, z\}_\Gamma$ is an algebraic function of w.

The genus zero case: *Hauptmoduln*

If $X(\Gamma)$ is of genus $g = 0$, then $\mathbb{C}(X(\Gamma)) = \mathbb{C}(w)$ where $w(z)$ is a Hauptmodul for $X(\Gamma)$.

Thus there is a rational function $R(w) \in \mathbb{C}(w)$ such that $w(z)$ is a solution of the third order differential equation

$$\{w, z\}_\Gamma = R(w).$$

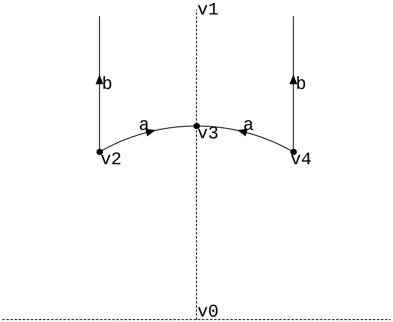
We can obtain $\zeta(w)$ by integrating the linear differential equation

$$\frac{d^2\eta}{dw^2} = R(w)\eta.$$

Remark

A key point is always the computation of $R(w)$.

How to obtain the *Hauptmodul* j?



$$\mathbf{SL}(2,\mathbb{Z})\backslash\mathcal{H}^* \xrightarrow{j(z)} \mathbb{P}^1(\mathbb{C})$$

$$\begin{aligned} j(z) &= \frac{1}{q} + 744 + 196\,884q + 21\,493\,7602^2 + O(q^3) \\ &= 1728\frac{g_2^3}{\Delta}, \qquad\qquad q = e^{2\pi i z}, \quad z \in \mathcal{H} \end{aligned}$$

Dedekind's *valence* function (1877)

$$[v,z] = \frac{-4}{\sqrt{\frac{dv}{dz}}}\frac{d^2}{dv^2}\sqrt{\frac{dv}{dz}} = 4\{z,v\}_{\mathbf{SL}(2,\mathbb{Z})}$$

$$v(i) = 1, \quad v(e^{\frac{2\pi i}{3}}) = 0, \quad v(\infty) = \infty$$

$$\frac{1}{(1-v)^{1/2}}\frac{dv}{dz}, \quad \frac{1}{v^{2/3}}\frac{dv}{dz}, \quad \frac{1}{v}\frac{dv}{dz}$$

Fuchs' theory

$$R(v) = \frac{3}{4(1-v)^2} + \frac{8}{9v^2} + \frac{23}{36(1-v)} + \frac{23b}{36v}$$

$$= \frac{36v^2 - 41v + 32}{36v^2(1-v)^2}$$

$$[v,z] = R(v), \quad \frac{d^2\eta}{dv^2} = -\frac{1}{4}R(v)\eta, \quad z(v) = \frac{\eta_1(v)}{\eta_2(v)}$$

Dedekind's valence function *versus* Klein's j invariant

The function

$$z(v) := \text{const.}\, v^{-\frac{1}{3}}(1-v)^{-\frac{1}{4}}\left(\frac{dv}{dz}\right)^{\frac{1}{2}}$$

satisfies the hypergeometric differential equation

$$v(1-v)\frac{d^2z}{dv^2} + \left(\frac{2}{3} - \frac{7v}{6}\right)\frac{dz}{dv} - \frac{z}{144} = 0$$

whose solutions are $c_1\eta_1(v) + c_2\eta_2(v)$, where

$$\eta_1(v) = F(1/12, 1/12, 2/3; v), \quad \eta_2(v) = F(1/12, 1/12, 1/2; 1-v)$$

$$\boxed{1728\, v(z) = j(z)}$$

Hypergeometric differential equation

$$\boxed{z(1-z)\frac{d^2w}{dz^2} + [c-(a+b+1)z]\,\frac{dw}{dz} - ab\,w = 0}$$

It has regular singular points at 0, 1, and infinity.
Its solutions are obtained in terms of the hypergeometric series

$$F(a,b,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!},\ |z|<1,\ \text{(Wallis,1655)}$$

where

$$(q)_n = \begin{cases} 1, & n=0 \\ q(q+1)\cdots(q+n-1), & n>0 \end{cases}$$

denotes de Pochhammer symbol.

Leo Pochhammer obtained his PhD in 1863 under Kummer in Berlin.

Outline

The Fermat curves F_N

$N \geqslant 4$ a positive integer

$$F_N : \quad X^N + Y^N = Z^N$$

$\deg(F_N) = N, \quad g(N) = (N-1)(N-2)/2$

$$\mathcal{D}_r = \{z \in \mathbb{C} : z\overline{z} < r^2\}$$

Δ a Fuchsian triangle group of signature (N, N, N) acting on $\mathcal{D}_r$:

$$\Delta = \langle \alpha, \beta, \gamma : \alpha^N = \beta^N = \gamma^N = \mathrm{Id},\ \alpha\beta\gamma = \mathrm{Id} \rangle$$

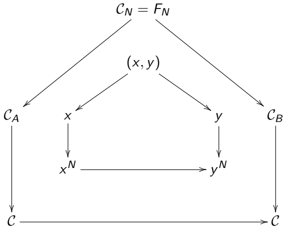
Theorem

A hyperbolic model for the Fermat curve F_N is given through an isomorphism

$$\Gamma \backslash \mathcal{D}_r^* \simeq F_N(\mathbb{C}),$$

where $\Gamma = [\Delta, \Delta]$ denotes the commutator subgroup of Δ.

First idea of the proof



$$g(\mathcal{C}) = g(\mathcal{C}_A) = g(\mathcal{C}_B) = 0$$

The Fermat curves as Riemann surfaces

We aim an explicit determination of affine coordinate functions $\mathrm{sf}(z;N), \mathrm{cf}(z;N)$, meromorphic on $\mathcal{D}_r^*$ and Γ-automorphic, realizing the isomorphism

$$\Gamma\backslash\mathcal{D}_r^* \simeq F_N(\mathbb{C}).$$

Thus, we shall have

$$\mathrm{sf}^N(z;N) + \mathrm{cf}^N(z;N) = 1, \quad \text{for all } z \in \Gamma\backslash\mathcal{D}_r^*,\ z \notin S,$$

for a certain finite subset S of $\Gamma\backslash\mathcal{D}_r^*$.

When it is not necessary to state the value of N explicitly, the *Fermat functions* $\mathrm{sf}(z;N), \mathrm{cf}(z;N)$ will be written $\mathrm{sf}(z), \mathrm{cf}(z)$.

The automorphism group of $\mathcal{D}_r$

Proposition

The group $\mathrm{Aut}(\mathcal{D}_r)$ consists of the following homographic transformations

$$f(z) = r^2 e^{i\alpha} \frac{z + z_0}{\bar{z}_0 z + r^2}, \quad z \in \mathcal{D}_r,$$

for $\alpha \in \mathbb{R}$ and $|z_0| < r$.

The group $\mathrm{Aut}(\mathcal{D}_r)$ also admits the equivalent description

$$\mathrm{Aut}(\mathcal{D}_r) \simeq \left\{ \begin{bmatrix} a & br \\ \overline{b}/r & \bar{a} \end{bmatrix} : a, b \in \mathbb{C},\ |b| < |a| \right\} / \mathbb{R}^*.$$

All groups $\mathrm{Aut}(\mathcal{D}_r)$ are conjugate in $\mathbf{PGL}(2, \mathbb{C})$. By considering the homothety $h_r(z) = rz$, we have

$$\mathrm{Aut}(\mathcal{D}_r) = h_r \mathrm{Aut}(\mathcal{D}_1) h_r^{-1}.$$

The triangle group Δ

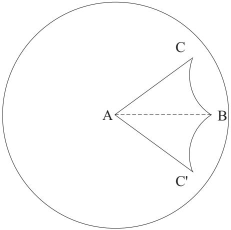
$\mathcal{T}_N = (A, B, C)$, $\mathcal{T}'_N = (A, B, C')$, interior angles $(\pi/N, \pi/N, \pi/N)$

α, β, γ rotations with center A, B, C and angle $2\pi/N$

$$\Delta = \langle \alpha, \beta, \gamma : \alpha^N = \beta^N = \gamma^N = \mathrm{Id},\ \alpha\beta\gamma = \mathrm{Id} \rangle$$

Proposition

The quadrilateral $\mathcal{Q} = AC'BC$ is a fundamental domain for the action of Δ in $\mathcal{D}_r$.
The quotient $\mathcal{C} = \Delta \backslash \mathcal{D}_r$ is a compact and connected Riemann surface of genus zero.



An involution

Proposition

Let $\zeta_N = e^{2\pi i/N}$ and $t = \sqrt{2\cos(\pi/N) - 1}$.

(i) The vertices of the triangles $\mathcal{T}_N, \mathcal{T}_N'$ are

$$A = 0, \quad B = r\,t, \quad C = \zeta_{2N} B, \quad C' = \overline{\zeta}_{2N} B.$$

(ii) The involution $\tau(z) = \dfrac{rz - rB}{(B/r)z - r}$ defined by the matrix

$$M(\tau) = \begin{bmatrix} ir & -iBr \\ iB/r & -ir \end{bmatrix}$$

is an element of $\mathrm{Aut}(\mathcal{D}_r)$ which interchanges the points A and B, and the points C and C'.

Coverings of degree N of $\mathcal{C} = \Delta \backslash \mathcal{D}_r$

$$\varphi_A : \Delta \longrightarrow \mathbb{Z}/N\mathbb{Z}, \quad \alpha \mapsto 1, \quad \beta \mapsto 0$$

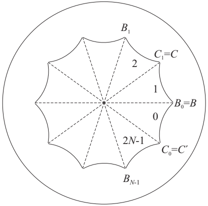
$$\Delta_A = \ker(\varphi_A) \text{ , subgroup generated by } \beta \text{ and } [\Delta, \Delta]$$

Proposition

(a) The hyperbolic regular polygon $\mathcal{P}_A = \cup_{i=0}^{N-1} \mathcal{Q}_i$, $\mathcal{Q}_i = \alpha^i(\mathcal{Q})$, is a fundamental domain for the action of Δ_A.

(b) The Riemann surface $\mathcal{C}_A = \Delta_A \backslash \mathcal{D}_r$ is a covering of degree N of $\mathcal{C}$ of genus zero.

(c) $\mathrm{Aut}(\mathcal{C}_A \mid \mathcal{C}) \simeq \Delta/\Delta_A = \langle \overline{\alpha} \rangle$, a cyclic group of order N.



Function fields for $\mathcal{C}_A$, $\mathcal{C}_B$

(d) There exists a Δ_A-automorphic function $\mathrm{sf}(z) = \mathrm{sf}(z; N)$, defined on $\mathcal{D}_r$, establishing an analytic isomorphism

$$\mathrm{sf} : \mathcal{C}_A = \Delta_A \backslash \mathcal{D}_r \longrightarrow \mathbb{P}^1(\mathbb{C})$$

and such that $\mathrm{sf}(A; N) = 0, \mathrm{sf}(B; N) = 1, \mathrm{sf}(C; N) = \infty$. The function field $\mathbb{C}(\mathcal{C}_A)$ is isomorphic to $\mathbb{C}(\mathrm{sf})$.

(e) There exists a Δ_B-automorphic function $\mathrm{cf}(z) = \mathrm{cf}(z; N)$, defined on $\mathcal{D}_r$, establishing an analytic isomorphism between

$$\mathrm{cf} : \mathcal{C}_B = \Delta_B \backslash \mathcal{D}_r \longrightarrow \mathbb{P}^1(\mathbb{C})$$

and such that $\mathrm{cf}(A; N) = 1, \mathrm{cf}(B; N) = 0, \mathrm{cf}(C; N) = \infty$. The function field $\mathbb{C}(\mathcal{C}_B)$ is isomorphic to $\mathbb{C}(\mathrm{cf})$.

Algebraic dependence of *sf* and *cf*

Proposition

Let τ be the involution which interchanges the points A and B, and the points C and C'. Then

(a) $\mathrm{sf} \circ \tau = \mathrm{cf}$.

(b) For some $r, s \in \mathbb{Z}$, coprime with N, we have

$$\mathrm{sf} \circ \alpha = \zeta_N^r \, \mathrm{sf}, \qquad \mathrm{cf} \circ \beta = \zeta_N^s \, \mathrm{cf}.$$

(c) $\mathbb{C}(\mathcal{C}) = \mathbb{C}(\mathrm{sf}^N) = \mathbb{C}(\mathrm{cf}^N)$.

(d) For any $z \in \mathcal{C}$, $z \neq C$, we have that

$$\mathrm{sf}^N(z) + \mathrm{cf}^N(z) = 1.$$

A fundamental domain for $[\Delta, \Delta]$

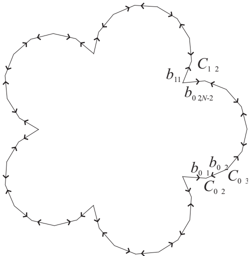
$$\varphi : \Delta \longrightarrow \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}, \quad \alpha \mapsto (1,0), \quad \beta \mapsto (0,1)$$
$$\ker(\varphi) = [\Delta, \Delta] =: \Gamma_N$$

Proposition

Let $\mathcal{C}_N := \Gamma_N \backslash \mathcal{D}_r$ and $H_N = \Delta/\Gamma_N = \mathrm{Aut}(\mathcal{C}_N \mid \mathcal{C})$. Then

(a) The elements $\{\beta_i^j \alpha^i\}_{i,j=0}^{N-1}$ represent the classes in H_N.

(b) Let $\mathcal{Q}_{i,j} := \beta_i^j \alpha^i(\mathcal{Q}) = \beta_i^j(\mathcal{Q}_i) = \alpha^i(\mathcal{Q}^j)$, then the polygon $\mathcal{P}_N = \cup_{i,j=0}^{N-1} \mathcal{Q}_{i,j}$ is a fundamental domain for Γ_N.



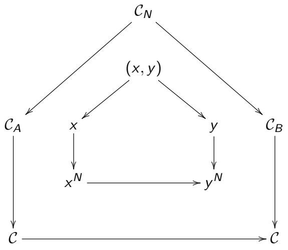
The function field of the Fermat curve

Proposition

We have that $\mathcal{C}_N \simeq F_N(\mathbb{C})$ as Riemann surfaces. Thus,

$$\mathbb{C}(F_N) = \mathbb{C}(\mathrm{sf}, \mathrm{cf}).$$

The functions $(\mathrm{sf}, \mathrm{cf})$ parametrize the Fermat curve F_N and are Γ_N-automorphic.



Computing the Schwarzian differential equation

$w = f(z)$

$$D_s(f(z), z) := \frac{2f'(z)f'''(z) - 3f''(z)^2}{f'(z)^2}$$

$$D_a(f(z), z) := \frac{D_s(f(z), z)}{f'(z)^2} = -D_s(f^{-1}(w), w)$$

Proposition

The isotropy groups of the points A, B, C under the action of Δ are $\langle\alpha\rangle, \langle\beta\rangle, \langle\gamma\rangle$, respectively. All of them are cyclic groups of order N.

$$\mathbb{C}(\mathcal{C}) = \mathbb{C}(\mathrm{sf}^N) = \mathbb{C}(\mathrm{cf}^N)$$

Computing the Schwarzian differential equation

Proposition

The function $\mathrm{sf}^N(z;N)$ is a solution of the differential equation

$$D_a(f(z),z) = -R(f(z)),$$

where

$$R(w) = \frac{1-1/N^2}{(w-w_A)^2} + \frac{1-1/N^2}{(w-w_B)^2} + \frac{m_A}{w-w_A} + \frac{m_B}{w-w_B};$$

and
$w_A = \mathrm{sf}^N(A) = 0$, $\mathrm{sf}^N(B) = w_B = 1$, $w_C = \mathrm{sf}^N(C) = \infty$,
m_A, m_B are two constants determined by the local conditions at the point C, where sf^N takes the value ∞:

$$\left.\begin{array}{l} m_A + m_B = 0 \\ m_A w_A + m_B w_B + 1 - 1/N^2 = 0 \end{array}\right\}.$$

Computing the Schwarzian differential equation

Thus, $m_A = -m_B = 1 - N^{-2}$ and

$$R(w) = \frac{(N-1)(N+1)(w^2 - w + 1)}{N^2(w-1)^2 w^2}.$$

Remark.

We have written the equation in terms of the values of sf^N at A, B to show how to write the equation corresponding to a different uniformizing parameter $\dfrac{a\,\mathrm{sf}^N + b}{c\,\mathrm{sf}^N + d}$, $ad - bc \neq 0$, for the curve $\mathcal{C}$.

Outline

Solving the Schwarzian equation

Since $sf^N(A)=0$, around the point $A=0$ the solutions of the differential equation will be of the shape

$$sf^N(z)=a_N z^N+a_{2N}z^{2N}+a_{3N}z^{3N}+a_{4N}z^{4N}+O(z^{5N}).$$

If $q(z):=z^N$, we have $D_a(q,z)=\frac{1-N^2}{N^2}q^{-2}$ and we must solve the system produced by

$$\begin{aligned}
D_a(f(q(z)),z) &= \frac{1-N^2}{N^2a_N^2}q^{-2}+\frac{4(-1+N^2)a_{2N}}{N^2a_N^3}q^{-1}\\
&+\frac{6((2-4N^2)a_{2N}^2+(-1+3N^2)a_Na_{3N})}{N^2a_N^4}\\
&+\frac{4(8(-1+4N^2)a_{2N}^3+9(1-5N^2)a_Na_{2N}a_{3N}+2(-1+7N^2)a_N^2a_{4N})}{N^2a_N^5}q\\
&+O(q^2)=-R(f(q)).
\end{aligned} \tag{1}$$

The general solution

We obtain in this way a parametric family of solutions $f(z) = f(z; a_N)$ whose coefficients are

$$
\begin{aligned}
a_{1N} &= a_N \\
a_{2N} &= -\frac{a_N^2}{2}, \\
a_{3N} &= \frac{(1+11N^2)a_N^3}{2^4(-1+2N)(1+2N)} \\
a_{4N} &= -\frac{3(1+N^2)a_N^4}{2^4(-1+2N)(1+2N)} \\
a_{5N} &= \frac{(-13-138N^2+1593N^4+718N^6)a_N^5}{2^8(-1+2N)^2(1+2N)^2(-1+4N)(1+4N)} \\
a_{6N} &= -\frac{15(-5+42N^2+87N^4+20N^6)a_N^6}{2^9(-1+2N)^2(1+2N)^2(-1+4N)(1+4N)} \\
&\vdots
\end{aligned}
$$

Taylor expansion of sf^N

By taking into account the initial condition $\mathrm{sf}^N(B) = 1$, we shall obtain a particular value λ_N of the parameter a_N for which

$$\mathrm{sf}^N(z; N) = \lambda_N z^N - \frac{\lambda_N^2}{2} z^{2N} + \frac{(1 + 11N^2)\lambda_N^3}{2^4(-1 + 2N)(1 + 2N)} z^{3N} + O(z^{4N}).$$

Now we extract the N^{th}-root of $f(z; a_N)$ to deduce a series expansion $g(z; b_1)$ around point A such that

$$g(z; b_1)^N = f(z; a_N) = \sum_{j \geq 1} a_{jN} z^{jN}. \qquad (2)$$

Taylor expansion of sf

We write

$$g(z;b_1)=\sum_{k\geq 0} b_{kN+1}z^{kN+1},$$

and deduce by substitution in equality (2) a linear system of equations for the coefficients b-s. When we solve it, we obtain

$$g(z;b_1)=$$

$$b_1 z-\frac{b_1^{N+1}}{2N}z^{N+1}+\frac{(-1+3N)(2+N)(1+N)b_1^{2N+1}}{2^4N^2(1+2N)(-1+2N)}z^{2N+1}+O(z^{3N+1}), \tag{3}$$

where $b_1^N=a_N$.

For a particular value μ_N of the parameter b_1, we shall have

$$\mathrm{sf}(z;N)=g(z;\mu_N),\quad \lambda_N=\mu_N^N.$$

Taylor expansion of cf

By performing analogous computations, we find the Taylor series around the point A of the function $\mathrm{cf}(z; N)$. Thus,

$$\mathrm{sf}(z; N) =$$

$$\mu_N z - \frac{1}{2N}(\mu_N z)^{N+1} + \frac{(-1+3N)(2+N)(1+N)}{2^4 N^2 (1+2N)(-1+2N)}(\mu_N z)^{2N+1} + O((\mu_N z)^{3N+1})$$

$$\mathrm{cf}(z; N) =$$

$$\sqrt[N]{1 - \mathrm{sf}^N(z)} = 1 - (\mu_N z)^N + \frac{1}{2N^2}(\mu_N z)^{2N} + O((\mu_N z)^{3N})$$

Remark. We have determined both coordinate functions sf, cf up to the *local constant* μ_N or the *local parameter* $q(z) := \mu_N z$.

Local constants

- The appearance of the local constant μ_N is not surprising because, up to now, we have not imposed the initial condition $\mathrm{sf}(B; N) = 1$.

- Since the function $\mathrm{sf}(z)$ has a pole at the point C, the point B lies on the boundary of the convergence disk of the series of the function around zero. Hence, in order to obtain a good numerical approximation of μ_N, it is not advisable to compute many terms of the series $\mathrm{sf}(z)$ and then impose $\mathrm{sf}(B) = 1$.

- The indeterminacy of μ_N reflects the random choice of the radius r of the disk $\mathcal{D}_r$ that we have taken to build up the fundamental domains for our curves.

- The indeterminacy of μ_N also reflects the random choice of the conjugacy class of the Fuchsian group $\Gamma = [\Delta, \Delta]$ used to uniformize the curve F_N.

Determining the inverse functions $\operatorname{arc}\mathrm{sf}$ and $\operatorname{arc}\mathrm{cf}$

The solutions of the differential equation

$$D_s(g(w), w) = R(w) \tag{4}$$

are the inverse functions of the solutions of the differential equation

$$D_a(f(z), z) = -R(f(z)) \tag{5}$$

(Poincaré) The solutions of (4) are quotients of two linearly independent solutions of the second order differential equation

$$u''(w) + \frac{1}{4}R(w)u(w) = 0. \tag{6}$$

Relation to the hypergeometric equations

The substitution $v(w) = s(w)u(w)$ transforms the equation

$$u''(w) + \frac{1}{4}R(w)u(w) = 0 \tag{7}$$

into an equation of the type

$$v''(w) + P(w)v'(w) + Q(w)v(w) = 0, \tag{8}$$

where

$$P(w) = -2\frac{d}{dw}\log s(w), \quad Q(z) = \frac{1}{4}R(w) + \frac{2s'(w)^2 - s(w)s''(w)}{s(w)^2}.$$

By a suitable election of $s(w)$, equation (8) turns out to be a hypergeometric equation. In our case, we take

$$s(w) = (w(w-1))^{\frac{1-N}{2N}}.$$

Relation to the hypergeometric functions

In this way we arrive at the hypergeometric equation

$$w(w-1)v''(w)+\frac{N-1}{N}(2w-1)v'(w)+\frac{(N-3)(N-1)}{4N^2}v(w)=0. \quad (9)$$

The general solution, $v(w)$, of equation (9) is

$$c_1F(\frac{N-1}{2N},\frac{N-3}{2N},\frac{N-1}{N};w)+c_2w^{\frac{1}{N}}F(\frac{N+1}{2N},\frac{N-1}{2N},\frac{N+1}{N};w),$$

where

$$F(a,b,c;w)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tw)^{-a}dt$$
$$=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{w^n}{n!} \quad (10)$$

is the well known hypergeometric function.

The $\mathrm{arc}\, f(z;1)$ computation

Proposition

Let us take $a_N = 1$. The inverse function $\mathrm{arc}\, f(z;1)$ of $f(z;1)$ is the quotient of two hypergeometric functions:

$$\mathrm{arc}\, f(z;1)(w) = w^{\frac{1}{N}} \frac{F(\frac{N+1}{2N}, \frac{N-1}{2N}, \frac{N+1}{N}; w)}{F(\frac{N-1}{2N}, \frac{N-3}{2N}, \frac{N-1}{N}; w)}$$

$$= w^{\frac{1}{N}} \left(1 + \frac{1}{2N} w + \frac{(N+1)(13N^2-5N-2)}{16N^2(2N+1)(2N-1)} w^2 \right. \quad (11)$$

$$\left. + \frac{(N+1)(23N^2-15N-2)}{96N^3(2N-1)} w^3 + \dots \right).$$

The result provides an easy way of computing the series $\mathrm{arc}\, f(z;1)$ and offers an alternative approach to the computation of $f(z;1)$.

Computing the local constant

From $1 = \mathrm{sf}^N(B; N) = f(B; \lambda_N) = f(\mu_N B; 1)$, we obtain

$$\mu_N B = \mathrm{arc}f(z; 1)(1) = \frac{F(\frac{N+1}{2N}, \frac{N-1}{2N}, \frac{N+1}{N}; 1)}{F(\frac{N-1}{2N}, \frac{N-3}{2N}, \frac{N-1}{N}; 1)}$$

Since

$$F(a, b, c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},$$

we obtain

$$\mu_N B = \frac{\Gamma(\frac{N+1}{N})\Gamma(\frac{N-1}{2N})}{\Gamma(\frac{N-1}{N})\Gamma(\frac{N+3}{2N})}.$$

Since the right-hand-side term is real and $B = r_N\sqrt{2\cos(\pi/N) - 1}$, we deduce that $\mu_N \in \mathbb{R}$ and

$$r_N \propto \mu_N^{-1}.$$

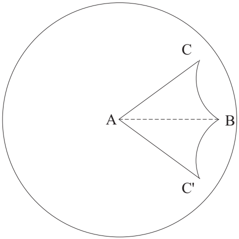
Choosing the point B

For each value of N there exists a special value of B which is the most natural one to parametrize F_N.

$B = \pi_N :=$ length of the one-dimensional simplex contained in $F_N(\mathbb{R})$ joining the points $(0, 1)$ and $(1, 0)$:

$$\pi_N := \int_0^1 \sqrt{1 + D(\mathrm{cf}(\mathrm{sf}),\, \mathrm{sf})^2}\; d(\mathrm{sf})$$

$$= \int_0^1 \sqrt{1 + \left(\frac{\mathrm{sf}}{\mathrm{cf}}\right)^{2N-2}}\; d(\mathrm{sf})$$



Determining the radius r_N and the local parameter q

Combining all these results, we obtain:

$$(\mathrm{sf}(A), \mathrm{cf}(A)) = (\mathrm{sf}(0), \mathrm{cf}(0)) = (0, 1)$$

$$(\mathrm{sf}(B), \mathrm{cf}(B)) = (\mathrm{sf}(\pi_N), \mathrm{cf}(\pi_N)) = (1, 0)$$

$$r_N = \frac{\pi_N}{\sqrt{2\cos(\pi/N) - 1}}$$

$$\mu_N = \frac{1}{\pi_N} \frac{\Gamma(\frac{N+1}{N})\Gamma(\frac{N-1}{2N})}{\Gamma(\frac{N-1}{N})\Gamma(\frac{N+3}{2N})}$$

$$q(z) = \mu_N z, \quad \text{local parameter}$$

Outline

1. Curves and Riemann surfaces
2. Fermat curves
3. The Fermat sinus and cosinus functions: $(\mathrm{sf}, \mathrm{cf})$
4. Fermat tables

N	π_N	r_N	μ_N
4	1.75442	2.72598	0.917155
5	1.79861	2.28787	0.835412
6	1.82943	2.13819	0.779984
7	1.85211	2.06822	0.740087
8	1.86949	2.03043	0.710054
9	1.88323	2.00823	0.686653
10	1.89435	1.99448	0.667917
11	1.90354	1.98568	0.652585
12	1.91127	1.97992	0.639808
13	1.91785	1.97613	0.629000
14	1.92352	1.97364	0.619738
15	1.92846	1.97203	0.611715
16	1.9328	1.97104	0.604697
17	1.93665	1.97049	0.598507
18	1.94007	1.97024	0.593007
19	1.94315	1.97021	0.588088
20	1.94593	1.97034	0.583662

Table 1. Values of π_N, r_N, μ_N

n	$\mathrm{sf}(z;4)$	n	$\mathrm{cf}(z;4)$
1	1	0	1
5	$-\dfrac{15}{5!}$	4	$-\dfrac{6}{4!}$
9	$\dfrac{7425}{9!}$	8	$\dfrac{1260}{8!}$
13	$-\dfrac{18822375}{13!}$	12	$-\dfrac{2316600}{12!}$
17	$\dfrac{159120014625}{17!}$	16	$\dfrac{15081066000}{16!}$
21	$-\dfrac{3416758559589375}{21!}$	20	$-\dfrac{261570317580000}{20!}$
25	$\dfrac{154667733894382190625}{25!}$	24	$\dfrac{9957261810295800000}{24!}$
29	$-\dfrac{13152597869424682778484375}{29!}$	28	$-\dfrac{729754600219383538800000}{28!}$

Table 2. Taylor coefficients of $\mathrm{sf}(z;4)$, $\mathrm{cf}(z;4)$ at 0, $(g=3)$

n	$\mathrm{sf}(z;37)$	n	$\mathrm{cf}(z;37)$
1	1	0	1
38	$-\frac{1}{74}$	37	$-\frac{1}{37}$
75	$\frac{2717}{1998740}$	74	$\frac{1}{2738}$
112	$-\frac{13091}{147906760}$	111	$-\frac{10739}{73953380}$
149	$\frac{744697343}{166669797434672}$	148	$\frac{21113}{5472550120}$
186	$-\frac{61959482923}{154169562627071600}$	185	$-\frac{43945156471}{77084781313535800}$
223	$\frac{563138467716575}{1485757975523610357228 8}$	222	$\frac{279990543017}{11408547634403298400}$
260	$-\frac{5081316514596887}{245415379885596353649400 0}$	259	$-\frac{2158310701054223}{858953829599587237772900}$
297	$\frac{21158496912821252478247}{18396943935707987680699411155840 0}$	296	$\frac{682866283188190333}{5085006671229556447615568000}$
334	$-\frac{111875905818620841368622437}{1014223519175581360836958537021459200 0}$	333	$-\frac{698788932021482486947390 21}{5071117595877906804184792685107296000}$
371	$\frac{16215411705290449514944498325753}{1984251572354073980154539381153183634387200 0}$	370	$\frac{70931865126243668483931994 7}{9381567552374127587741866467448497600 00}$

Table 3. Reduced Taylor coefficients of $\mathrm{sf}(z;37)$, $\mathrm{cf}(z;37)$ at 0, $(g = 630)$